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On the parallel transport of the Ricci curvatures

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Abstract

Geometrical characterizations are given for the tensor $R \cdot S$, where S is the *Ricci tensor* of a (semi-)Riemannian manifold (M, g) and R denotes the *curvature operator* acting on S as a derivation, and of the *Ricci Tachibana tensor* $\wedge_g \cdot S$, where the natural *metrical operator* $\wedge_g$ also acts as a derivation on S. As a combination, the *Ricci curvatures* associated with directions on M, of which the isotropy determines that M is *Einstein*, are extended to the *Ricci curvatures of Deszcz* associated with directions and planes on M, and of which the isotropy determines that M is *Ricci pseudo-symmetric in the sense of Deszcz*.

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1. Introduction

Recently, the *parallel transport of sectional curvatures* K on a (semi-)Riemannian manifold (M, g) around infinitesimal coordinate parallelograms was studied in [23]. There, amongst other things, new geometrical interpretations of the (0, 6)-curvature tensor $R \cdot R$ on M were obtained, where the first R stands for the *curvature operator* acting as a derivation on the second R which stands for the (0, 4)-*curvature tensor of Riemann and Christoffel*, as well as of the (0, 6)-Tachibana tensor $Q(g, R) := \wedge_g \cdot R$, where the classical metrical endomorphism $\wedge_g$ also acts on the (0, 4)-curvature tensor R as a derivation. By comparison of these (0, 6)-tensors $R \cdot R$ and $Q(g, R)$, a *new scalar valued Riemannian curvature invariant* was determined on (M, g), the so-called *double sectional curvature* or the *sectional curvature of Deszcz*, $L(p, \pi, \overline{\pi})$, which depends on two tangent 2-planes π and $\overline{\pi}$ at any point p of M. And, the manifolds (M, g) for which the sectional curvature of Deszcz is *isotropic*, i.e., does not depend on the planes π and $\overline{\pi}$ at p, but remains a scalar valued function which at most depends only on the points of

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M, say $L : M \to \mathbb{R} : p \mapsto L(p)$, are the manifolds which are *pseudo-symmetric in the sense of Deszcz* (see e.g. [13,22,31]). In particular, we point out that, for instance, the classical spacetimes, such as the Schwarzschild and Friedman–Lemaître–Robertson–Walker models, are all pseudo-symmetric [11,17,21]. Further, the three-dimensional *Thurston geometries* and the three-dimensional *d'Atri spaces* are models of pseudo-symmetric spaces of *constant type* [24,25], i.e., pseudo-symmetric spaces for which the Riemann curvature L of Deszcz is constant: 0 (for the semi-symmetric spaces), $+1$ (for the special linear group, the Heisenberg group and the special unitary group) and -1 (for the Lie group Sol).

In the present article, we basically carry out a similar study for the (0, 2)-*Ricci tensor* S instead of the (0, 4)-Riemann–Christoffel curvature tensor. New geometrical interpretations of the (0, 4)-tensor $R \cdot S$ and of the *Ricci Tachibana tensor* $Q(g, S) = \wedge_g \cdot S$ are given, in particular, thus characterizing the *Ricci semi-symmetric spaces* ($R \cdot S = 0$) and the Einstein spaces ($S = \lambda\, g$, $\lambda \in \mathbb{R}$). Then the so-called *Ricci curvature of Deszcz*, $L_S(p, d, \overline{\pi})$, is defined. This is a new scalar valued Riemannian invariant curvature function depending on a tangent direction d and a tangent plane $\overline{\pi}$ at any point p of M. This curvature is *isotropic*, i.e., this $L_S(p, d, \overline{\pi})$ is actually a scalar valued function on M a priori only depending on the points p of M, when the manifold M is *Ricci pseudo-symmetric* in the sense of Deszcz. In [23], in the same way as one may make the step from *locally flat* spaces ($R = 0$, $K = 0$) to the real space forms, i.e., to the spaces of constant sectional curvature $K = c$ ($=, >, <0$), the step is made from the *semi-symmetric spaces of Szabó* [29,30] ($R \cdot R = 0$, $L = 0$) to the *pseudo-symmetric spaces of Deszcz* in terms of the scalar valued curvature functions $L = L(p)$ (it should be stressed that, for L, there is no equivalent version of the lemma of Schur which holds for K; see e.g. [13]). Similarly, in Section 3, we go from the *Einstein spaces*, i.e., the spaces of constant Ricci curvatures, to the spaces of *isotropic Ricci curvatures of Deszcz*. At the end, some non-trivial examples of these kinds of manifolds are given.

2. A geometrical interpretation of the tensor $R \cdot S$

Let (M, g) be an n-dimensional *Riemannian manifold* with *metric* g. Denote the *Levi-Civita connection* by ∇ and its related *Riemann–Christoffel* $(1, 3)$-curvature tensor by R. The endomorphisms $V \wedge_g W$ and $R(V, W)$ of the Lie algebra of vector fields $\mathfrak{X}(M)$ of M are defined by

$$\big(V \wedge_g W\big)\, Z = g(W, Z)V - g(V, Z)W,$$

and

$$R(V, W)Z = \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V,W]} Z;$$

(here and in the following vectors will be systematically denoted by lower case letters, while vector fields will be denoted by capital letters). As is well known, following Schouten [28], $R(x, y)z$ measures the second-order change of a vector $z \in T_pM$, $p \in M$; namely after the *parallel transport of* z *around an infinitesimal coordinate parallelogram* $\mathcal{P}$ *cornered at* p with sides of lengths Δx and Δy tangent to x and y at p, a vector $z^\star \in T_pM$ is obtained,

$$z^\star = z + [R(x, y)z]\, \Delta x \Delta y + O^{>2}(\Delta x, \Delta y).$$

And a vector $(x \wedge_g y)z$ can be geometrically interpreted as follows. Assume that $x, y \in T_pM$ are orthonormal and choose $\{e_3, \ldots, e_n\}$ so that $\{x, y, e_3, \ldots, e_n\}$ is an orthonormal basis of T_pM. Then $z \in T_pM$ can be decomposed by orthonormal expansion as

$$z = g(z, x)x + g(z, y)y + \sum_{i=3}^{n} g(z, e_i)e_i.$$

By rotating the projection of z onto the plane $x \wedge y$ spanned by x and y over an infinitesimal angle ε, while keeping the projection of z onto the $(n-2)$-plane spanned by $e_3, \ldots, e_n$ fixed, a new vector $\widetilde{z}$ is obtained, namely,

$$\widetilde{z} = z + \varepsilon\{g(z, y)x - g(z, x)y\} + O(\varepsilon^2).$$

Thus the vector $(x \wedge_g y)z$ measures the first-order change of the vector z after such an *infinitesimal rotation* of z in the plane $x \wedge y$ at the point p [23]. Therefore, it seems natural to consider $(x \wedge_g y)z$ as some kind of normalization for $R(x, y)z$, which gives a new interpretation of the following classical definition.

Definition 1. At any point $p \in M$, let $\pi = x \wedge y$ be any plane tangent to M at p, spanned by any two of its vectors x and y. Then, the real number

$$K(p, \pi) = \frac{g(R(x, y)y, x)}{g((x \wedge_g y)y, x)}$$

only depends on the point p and the plane π, and is called the sectional curvature of M at p for the plane section π.

Putting $G = \frac{1}{2} g \wedge g$, where $\wedge$ is the Nomizu–Kulkarni product of (0, 2)-forms, we have $G(x, y, x, y) = g((x \wedge_g y)x, y)$. The (0, 4)-curvature tensor R is related to the $(1, 3)$-curvature tensor by $R(X, Y, Z, W) = g(R(X, Y)Z, W)$. Thus we obtain the familiar expression

$$K(p, \pi) = \frac{R(x, y, y, x)}{G(x, y, y, x)}.$$

The *Ricci tensor* S is defined as the trace of the Riemann–Christoffel curvature tensor,

$$S(X, Y) = \sum_{i=1}^{n} R(E_i, X, Y, E_i),$$

with $X, Y \in \mathfrak{X}(M)$ and $\{E_1, E_2, \ldots, E_n\}$ any orthonormal basis of $\mathfrak{X}(M)$. At a point $p \in M$, let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of $T_p M$. The Ricci curvature in the direction of e_1, $\mathrm{Ric}(e_1)$, can then be written as

$$\mathrm{Ric}(e_1) = \sum_{j=2}^{n} K(p, e_1 \wedge e_j).$$

Next we give a geometrical interpretation of the (0, 4)-tensor $R \cdot S$, obtained by the action of the curvature operator $R(X, Y)$ on the (0, 2)-symmetric Ricci tensor,

$$\begin{aligned}(R \cdot S)(X_1, X_2; X, Y) &:= (R(X, Y) \cdot S)\,(X_1, X_2) \\ &= -S\,(R(X, Y)X_1, X_2) - S\,(X_1, R(X, Y)X_2)\,,\end{aligned}$$

where $X_1, X_2, X, Y \in \mathfrak{X}(M)$. A Riemannian manifold is said to be *Ricci semi-symmetric* when $R \cdot S$ vanishes identically, i.e., $R \cdot S = 0$. Now, consider a vector v at any point $p \in M$ and any coordinate parallelogram $\mathcal{P}$ cornered at p with sides of lengths Δx and Δy tangent to the linearly independent vectors x and y at p. Then, by *parallel transport* of v around $\mathcal{P}$ we obtain the vector $v^\star = v + [R(x, y)v]\Delta x \Delta y + O^{>2}(\Delta x, \Delta y)$, so that

$$\mathrm{Ric}(v^\star) = \mathrm{Ric}(v) - [(R \cdot S)(v, v; x, y)]\,\Delta x \Delta y + O^{>2}(\Delta x, \Delta y).$$

Theorem 2. *Let $\mathcal{P}$ be any infinitesimal coordinate parallelogram cornered at a point p of a Riemannian manifold M with sides of lengths Δx and Δy, which are tangent to vectors x and y at p. Let $v^\star$ be the vector obtained from v after parallel transport along $\mathcal{P}$. Then, in second-order approximation,*

$$\delta_{\mathcal{P}} Ric(v) = -(R \cdot S)(v, v; x, y)\Delta x \Delta y,$$

i.e., the (0, 4)*-tensor $R \cdot S$ of M measures the change in Ricci curvature at any point p for any vector v under parallel transport of v around any infinitesimal coordinate parallelogram $\mathcal{P}$ cornered at p.*

Corollary 3. *A Riemannian manifold M is Ricci semi-symmetric if and only if its Ricci curvature function is invariant, up to second order, under parallel transport of any vector v at any point p of M around any infinitesimal coordinate parallelogram $\mathcal{P}$ cornered at p.*

Every semi-symmetric manifold is Ricci semi-symmetric. The converse however is not true in general. After Defever proved the existence of Ricci semi-symmetric, not semi-symmetric hypersurfaces in Euclidean spaces, an explicit example was given by Abdalla and Dillen in [1]; a complete intrinsic classification of Ricci semi-symmetric hypersurfaces of Euclidean space was presented by Mirzoyan in [27].

As can easily be seen, the tensor $R \cdot S$ has the following algebraic symmetry properties:

$$(R \cdot S)(X_1, X_2; X, Y) = (R \cdot S)(X_2, X_1; X, Y) = -(R \cdot S)(X_1, X_2; Y, X).$$

The simplest (0, 4)-tensor on an n-dimensional Riemannian manifold having the same symmetry properties as $R \cdot S$ may well be the *Ricci–Tachibana tensor* $Q(g, S)$ defined by

$$\begin{aligned} Q(g, S)(X_1, X_2; X, Y) &:= \big((X \wedge_g Y) \cdot S\big)(X_1, X_2) \\ &= -S\big((X \wedge_g Y)X_1, X_2\big) - S\big(X_1, (X \wedge_g Y)X_2\big). \end{aligned}$$

Lemma 4. *A Riemannian manifold* (M, g) *is Einstein if and only if* $Q(g, S) = 0$.

Proof. If (M, g) is Einstein, i.e., if $S = \lambda\, g$, then it follows straightforwardly that $Q(g, S)$ vanishes identically. Assume, conversely, that $Q(g, S)(X_1, X_2; X, Y) = 0$, for all $X_1, X_2, X, Y \in \mathfrak{X}(M)$. Because M is Riemannian, there exists a basis $\{E_1, E_2, \ldots, E_n\}$ which diagonalizes S, i.e., $S(E_i, E_j) = \lambda_i\, g(E_i, E_j)$, $i, j \in \{1, \ldots, n\}$, where $\lambda_1, \ldots, \lambda_n$ are the eigenfunctions of the Ricci tensor. Then, for $i \neq j$, $Q(g, S)(E_i, E_j; E_i, E_j) = \lambda_j - \lambda_i = 0$, and hence (M, g) is Einstein. □

Using the above geometrical interpretation of $(x \wedge_g y)z$, we now give a geometrical meaning to the components $Q(g, S)(v, v; x, y)$ of the Ricci–Tachibana tensor. Let $\{x, y, e_3, \ldots, e_n\}$ be an orthonormal basis of T_pM and consider a vector $v \in T_pM$. The vector $\widetilde{v}$ is the vector obtained after an infinitesimal rotation of the projection of v in the plane span$\{x, y\}$ as before, namely

$$\widetilde{v} = v + \varepsilon(x \wedge_g y)v + O(\varepsilon^2).$$

Comparing the Ricci curvatures $\mathrm{Ric}(v)$ and $\mathrm{Ric}(\widetilde{v})$ we find

$$\mathrm{Ric}(\widetilde{v}) = \mathrm{Ric}(v) - \varepsilon Q(g, S)(v, v; x, y) + O(\varepsilon^2).$$

The components $Q(g, S)(v, v; x, y)$ thus measure the change of the Ricci curvature $\mathrm{Ric}(v)$ under an operation involving infinitesimal rotations performed at a point p, without leaving this point, as described above. In contrast, the components $(R \cdot S)(v, v; x, y)$ measure the change of Ricci curvature $\mathrm{Ric}(v)$ after a parallel translation of the vector v in an infinitesimal neighborhood of the point p. It seems therefore natural to consider the components $Q(g, S)(v, v; x, y)$ as some kind of normalization for the components $(R \cdot S)(v, v; x, y)$. In analogy with Definition 1 we propose the following.

Definition 5. Let (M, g) be an $n(\geq 3)$-dimensional Riemannian manifold which is not Einstein. Let $\mathcal{U}$ be the set of points where the Ricci–Tachibana tensor $Q(g, S)$ is not identically zero, i.e., $\mathcal{U} = \{x \in M \mid Q(g, S)_x \neq 0\}$; assume that M is not Einstein such that $\mathcal{U} \neq \emptyset$. Then, at a point $p \in \mathcal{U}$, a direction d, spanned by a vector $v \in T_pM$, is said to be curvature dependent on a plane $\overline{\pi} = x \wedge y \subset T_pM$ if $Q(g, S)(v, v; x, y) \neq 0$ (which condition, of course, is independent of the choice of basis $\{x, y\}$ for $\overline{\pi}$ and of vector v which determines the direction d).

Definition 6. At a point $p \in \mathcal{U}$, let d be a direction, spanned by a vector v, which is curvature dependent on a plane $\overline{\pi} = x \wedge y$. We define the Ricci curvature $L_S(p, d, \overline{\pi})$ of Deszcz of the direction d and the plane $\overline{\pi}$ as the scalar

$$L_S(p, d, \overline{\pi}) = \frac{(R \cdot S)(v, v; x, y)}{Q(g, S)(v, v; x, y)},$$

(which expression is independent of the choice of basis $\{x, y\}$ for the plane $\overline{\pi}$ as well as of the vector v which determines the direction d).

3. Properties of the Ricci curvature of Deszcz

We first give a geometrical interpretation of the Ricci curvature of Deszcz L_S in terms of the *parallelogramoïds constructed* as follows *by Levi-Civita* [6,20,26]. Let u and v be any two independent tangent vectors at any point p of M. Consider through p the geodesic α with tangent u and let q be the point on this geodesic at an infinitesimal distance A from p. Denote by $v^\star$ the vector obtained after parallel transport of v from p to q along α. Then, through p and q consider the geodesics β_p and β_q which are tangent to v and $v^\star$, respectively. Fix on them the points $\overline{p}$ and $\overline{q}$ at the same infinitesimal distance B from p and q respectively. The parallelogramoïd cornered at p with sides tangent to u and v is then completed by the geodesic $\overline{\alpha}$ through $\overline{p}$ and $\overline{q}$. Let A' be the geodesic distance between $\overline{p}$ and $\overline{q}$.

Levi-Civita then showed that, in first-order approximation, the sectional curvature of the plane $\pi = u \wedge v$ can be expressed as

$$K(p, \pi) = \frac{A^2 - A'^2}{A^2 B^2 G(u, v, v, u)}, \tag{1}$$

where $G(u, v, v, u) = \sin^2(\psi)$, with ψ the angle between the vectors u and v.

Consider at $p \in M$ an orthonormal basis $\{v = e_1, e_2, \ldots, e_n\}$ of $T_p M$ and construct for every plane $v \wedge e_j$ the squaroïd of Levi-Civita (i.e., the parallelogramoïd with equal sides $A = B = \varepsilon$). Let ε'_j denote the length of the completing geodesic, in each of these squaroïds. The Ricci curvature $\mathrm{Ric}(v)$ can then, up to first approximation, be expressed as

$$\mathrm{Ric}(v) = \sum_{j=2}^{n} \frac{\varepsilon^2 - \varepsilon'^2_j}{\varepsilon^4}.$$

Next consider at p a plane $\overline{\pi} = x \wedge y$ and parallel transport the frame $\{v, e_2, \ldots, e_n\}$ around the infinitesimal coordinate parallelogram formed by the tangent vectors x and y at p. Further, construct the $2(n-1)$ squaroïds starting from the planes $v \wedge e_j$ and $v^\star \wedge e^\star_j$ ($j = 2, \ldots, n$), respectively, all with equal sides ε. In general, the lengths of the closing geodesics, ε'_j and $\varepsilon^{\star\prime}_j$, will be different. More precisely, we find, up to second order with respect to the sides Δx and Δy of the coordinate parallelogram, that

$$L_S(p, v, \overline{\pi}) = \frac{\sum_{j=2}^{n} \left(\varepsilon^{\star\prime 2}_j - \varepsilon'^2_j \right)}{\varepsilon^4 Q(g, S)(v, v; x, y)}.$$

Thus, the scalar $L_S(p, v, \overline{\pi})$ basically measures the difference of the sum of the lengths of the closing geodesics of the $2(n-1)$ squaroïds constructed from v, which are related by parallel transport of their generating frame around an infinitesimal parallelogram with tangent plane $\overline{\pi}$ at p.

Similar to the equivalence of information contained in the Riemann–Christoffel curvature tensor R on the one hand and in the sectional curvatures $K(p, \pi)$ on the other one, as was shown by Cartan [6], there is also an equivalence of information contained in the curvature tensor $R \cdot S$ of any Riemannian manifold (M, g) and in the Ricci curvatures $L_S(p, d, \overline{\pi})$ of Deszcz of directions d at p which are curvature dependent on planes $\overline{\pi} = x \wedge y$ at p, as shown in the following.

Theorem 7. *At any point $p \in \mathcal{U} \subset M$, the tensor $R \cdot S$ is completely determined by the Ricci curvatures L_S of Deszcz.*

Proof. Assume there exists a (0, 4)-tensor W with the same algebraic symmetries as $R \cdot S$ and such that for every vector v at $p \in \mathcal{U}$ which is curvature dependent on a plane $\overline{\pi} = x \wedge y$ at p,

$$\frac{(R \cdot S)(v, v; x, y)}{Q(g, S)(v, v; x, y)} = \frac{W(v, v; x; y)}{Q(g, S)(v, v; x, y)}.$$

Then, we have to prove that for all $x_1, x_2, x_3, x_4 \in T_p M$,

$$(R \cdot S)(x_1, x_2; x_3, x_4) = W(x_1, x_2; x_3, x_4).$$

Let T be the (0, 4)-tensor given by $T = R \cdot S - W$. Obviously, T has the same algebraic symmetries as $R \cdot S$ and W and so, for every pair of curvature-dependent vector v and plane $\overline{\pi} = x \wedge y$,

$$T(v, v; x, y) = 0\,. \tag{2}$$

When a vector v is not curvature dependent with respect to a plane $\overline{\pi}$, we have that $Q(g, S)(v, v; x, y) = 0$. Because this is a polynomial in the components of v, x and y, the zero set does not contain any open subset (for otherwise $Q(g, S)_p \equiv 0$, which would be a contradiction with $p \in \mathcal{U}$). So, we can choose series of tangent vectors $v_n \to v$, $x_n \to x$ and $y_n \to y$ such that for any n, v_n is curvature dependent on $x_n \wedge y_n$. Hence, for every n, $T(v_n, v_n; x_n, y_n) = 0$, and thus also in the limit $T(v, v; x, y) = 0$. Thus (2) holds for all $v, x, y \in T_p M$. Using polarization and the symmetry properties of $R \cdot S$ then completes the proof. □

Corollary 8. *The Ricci semi-symmetric spaces are characterized by the vanishing of their Ricci curvatures of Deszcz.*

Definition 9. A Riemannian manifold M of dimension n (≥ 3) is said to be Ricci pseudo-symmetric in the sense of Deszcz, at a point $p \in \mathcal{U}$, if there exists a scalar $L_S(p)$ such that, at p,

$$R \cdot S = L_S(p)\, Q(g, S).$$

A Riemannian manifold M of dimension n (≥ 3) is called Ricci pseudo-symmetric in the sense of Deszcz if it is Ricci pseudo-symmetric at all of its points.

Theorem 10. *A Riemannian manifold M of dimension n (≥ 3) is Ricci pseudo-symmetric in the sense of Deszcz if and only if at all of its points $p \in \mathcal{U}$ all the Ricci curvatures of Deszcz are the same, i.e., for all curvature-dependent directions d with respect to planes $\overline{\pi}$, $L_S(p, d, \overline{\pi}) = L_S(p)$.*

Proof. If $R \cdot S = L_s(p)\, Q(g, S)$ at p, the statement is obvious. Conversely, assume that $L_S(p, v, \overline{\pi}) = L_S(p)$ for any vector v curvature dependent on a plane $\overline{\pi} = x \wedge y$. Then,

$$(R \cdot S)(v, v; x, y) = L_S(p)\, Q(g, S)(v, v; x, y).$$

The tensor $T = R \cdot S - L_S(p)\, Q(g, S)$ has the same algebraic symmetries as $R \cdot S$. For a vector v which is curvature dependent on a plane $\overline{\pi} = x \wedge y$, one has $T(v, v; x, y) = 0$. If both are not curvature dependent, an argument as in the proof of Theorem 7 shows that $T(x_1, x_1; x_2, x_3) = 0$, for all $x_1, x_2, x_3 \in T_p M$, from which follows the result. □

Remarks and comments 11. If (M, g) is pseudo-symmetric, i.e., there exists a function $L_R : M \to \mathbb{R}$ such that $R \cdot R = L_R\, Q(g, R)$, then the manifold (M, g) is automatically Ricci pseudo-symmetric and the Ricci curvature of Deszcz L_S is equal to the sectional curvature of Deszcz L_R.

Whereas every pseudo-symmetric manifold thus automatically is Ricci pseudo-symmetric too, *the converse is not true*. For instance, every warped product manifold $M_1 \times_f M_2$ of a one-dimensional manifold (M_1, g_1) and a non-pseudo-symmetric (or say, generic) Einstein manifold (M_2, g_2) of dimension ≥ 3 is a non-pseudo-symmetric, Ricci pseudo-symmetric manifold [13]. Further examples of Ricci pseudo-symmetric, non-pseudo-symmetric spaces with in general *non-constant curvature function L_S* were given, amongst others, in [16,18,19]. Concerning Ricci pseudo-symmetric manifolds of *constant Ricci curvature of Deszcz*, we mention the following. In his work on *isoparametric hypersurfaces*, Cartan specifically studied the compact minimal hypersurfaces M^n of spheres $S^{n+1}(1)$, and among them met hypersurfaces with principal curvatures $-\sqrt{3}$, 0 and $\sqrt{3}$, all of the same multiplicity, which are called the *Cartan hypersurfaces of spheres* and which exist only when $n = 3$, 6, 12 and 24 [4,5]. For $n = 3$, such hypersurfaces clearly are quasi-Einstein and hence pseudo-symmetric (see e.g. [2], also in connection with the pseudo-symmetry of three-dimensional d'Atri spaces and the three-dimensional Thurston models which were mentioned before). However for $n = 6$, 12 and 24, the Cartan hypersurfaces are non-pseudo-symmetric, Ricci pseudo-symmetric manifolds with $L_S = 1$. For some more information on various pseudo-symmetries of hypersurfaces, see also [12,14,15,31].

A short and light historical note 12. The notation S for the Ricci tensor of a Riemannian manifold seems to be rather well chosen. Similarly to the *second fundamental form* of a submanifold usually being denoted by h, probably since its *first* or *metrical fundamental form* is gravitationally often denoted by g, there may have been consideration of writing S for the *Ricci tensor* of a Riemannian manifold and, incidentally, next writing τ for the *scalar curvature* of this manifold, since its *curvature tensor of Riemann and Christoffel* is written as R. But there is also the following. In his 1871 master thesis at the University of Kazan, entitled "On the characterization of three-dimensional systems" (in Russian; a French sort of summary appeared in 1873 in the Bulletin des Sciences Mathématiques), Suvorov was the first to explicitly undertake the evidently natural study of the *extremal values of the sectional curvature function* K on a Riemannian manifold, albeit only for the easiest of the non-trivial situations, namely for *the three-dimensional case*. In this case, of course, critical planes π for K correspond one-to-one to (their perpendicular) directions d, and thus these critical curvatures K, called the *principal curvatures* by Ricci in 1899 in his "Societá dei XL — vol XII" paper "On continuous groups of motions" (in Italian), correspond to the critical values of the *Ricci curvatures*, or equivalently, to the *eigenfunctions of the Ricci operator*. The above observations were taken from L. Bianchi's "Lezioni di geometria differenziale", vol 1 (section 162). In this connection, we would like to further observe that, in our opinion, whereas direct and general studies on the geometry of Riemannian manifolds with respect to min K and max K are rare (see e.g. Berger's panoramic view of Riemannian geometry [3]), B.Y. Chen's recent idea of studying min K and max K generally, but indirectly, as "incorporated" in his so-called δ-curvatures, $\delta(2) = \tau - \min K$ and $\widehat{\delta}(2) = \tau - \max K$, and via the isometric imbedding *theorem of Nash*, through the so-called general and optimal Chen and related inequalities and corresponding "ideal" submanifolds, made it possible to obtain several new, general and significant results of extrinsic *and* of intrinsic nature in this direction [7–10]. Whereas the more sophisticated δ-curvatures $\delta(n_1, n_2, \ldots, n_k)$ and $\widehat{\delta}(n_1, n_2, \ldots, n_k)$ ($n_i \geq 2$, $i \in \{1, \ldots, k\}$, $k \in \{1, \ldots, n\}$, $\sum_i n_i \leq n$) basically can be seen as generalizations of Ricci curvatures for say "multiple" directions on arbitrary dimensional Riemannian manifolds, the old master thesis of Suvorov looked at with hindsight, in the light of the new Chen curvatures, provided the natural step from the extremal values of the sectional curvatures K of three-dimensional Riemannian manifolds to the Ricci curvatures and the Ricci tensor S.

References

[1] B.E. Abdalla, F. Dillen, A Ricci-semi-symmetric hypersurface of Euclidean space which is not semi-symmetric, Proc. Amer. Math. Soc. 130 (2001) 1805–1808.

[2] M. Belkhelfa, R. Deszcz, L. Verstraelen, Symmetry properties of 3-dimensional d'Atri spaces, Kyungpook Math. J. 46 (2006) 367–376.

[3] M. Berger, A Panoramic View of Riemannian Geometry, Springer-Verlag, Berlin, 2003.

[4] É. Cartan, Familes de surfaces isoparametriques dans les espaces á curvature constante, Ann. Math. Pure Appl. 17 (1938) 177–191.

[5] É. Cartan, Sur des familes remarquables d'hypersurfaces isoparametriques dans les espaces physiques, Math. Z. 45 (1939) 313–319.

[6] É. Cartan, Leçons sur la géométrie des espaces de Riemann, Gauthier-Villars, Paris, 1963.

[7] B.Y. Chen, Some pinching and classification theorems for minimal submanifolds, Archiv. Math. 60 (1993) 568–578.

[8] B.Y. Chen, Riemannian Submanifolds, in: F. Dillen, L. Verstraelen (Eds.), Handbook of Differential Geometry, vol. 1, North-Holland, Elsevier, Amsterdam, 2000 (Chapter 3).

[9] B.Y. Chen, What can we do with Nash's embedding theorem? Soochow J. Math. 30 (2004) 303–338 (special issue in honor of Professor Bang-Yen Chen).

[10] B.Y. Chen, Strings of Riemannian invariants, inequalities, ideal immersions and their applications, in: Proc. Third Pacific Rim Geometry Conference, Intern. Press, Cambridge, MA, 1998, pp. 7–60.

[11] F. Defever, R. Deszcz, L. Verstraelen, L. Vrancken, On pseudosymmetric space-times, J. Math. Phys. 35 (1994) 5908–5921.

[12] F. Defever, R. Deszcz, P. Dhooghe, L. Verstraelen, S. Yaprak, On Ricci pseudosymmetric hypersurfaces in spaces of constant curvature, Results Math. 27 (1995) 227–236.

[13] R. Deszcz, On pseudo-symmetric spaces, Bull. Soc. Math. Belg. A 44 (1992) 1–34.

[14] R. Deszcz, Pseudosymmetric hypersurfaces in spaces of constant curvature, Tensor (N.S.) 58 (1997) 253–269.

[15] R. Deszcz, M. Hotloś, Z. Sentürk, On Ricci pseudosymmetric hypersurfaces in space forms, Demonstratio Math. 34 (2004) 203–214.

[16] R. Deszcz, P. Verheyen, L. Verstraelen, On some generalized Einstein metric conditions, Publ. Inst. Math. (Beograd) (N.S.) 60 (74) (1996) 108–120.

[17] R. Deszcz, L. Verstraelen, L. Vrancken, The symmetry of warped product space-times, Gen. Relativity Gravitation 23 (1991) 671–681.

[18] R. Deszcz, L. Verstraelen, S. Yaprak, Pseudosymmetric hypersurfaces in 4-dimensional spaces of constant curvature, Bull. Inst. Math. Acad. Sinica 22 (1994) 167–179.

[19] R. Deszcz, S. Yaprak, Curvature properties of Cartan hypersurfaces, Colloq. Math. 67 (1994) 91–98.

[20] L. Eisenhart, Riemannian Geometry, Princeton University Press, Princeton, 1997.

[21] S. Haesen, L. Verstraelen, Classification of the pseudosymmetric spacetimes, J. Math. Phys. 45 (2004) 2343–2346.

[22] S. Haesen, L. Verstraelen, Curvature and symmetries of parallel transport, in: M. Boucetta, J-M. Morvan (Eds.), Differential Geometry and Topology, Discrete and Computational Geometry, in: NATO Science Series, IOS Press, 2005, pp. 197–238 (Chapter 8).

[23] S. Haesen, L. Verstraelen, Properties of a scalar curvature invariant depending on two planes, Manuscripta Math. 122 (2007) 59–72.

[24] O. Kowalski, M. Sekizawa, Pseudo-symmetric spaces of constant type in dimension three-elliptic spaces, Rend. Mat., Serie VII 17 (1997) 477–512.

[25] O. Kowalski, M. Sekizawa, Pseudo-symmetric spaces of constant type in dimension three — non-elliptic spaces, Bull. Tokyo Gakugei Univ. (4) 50 (1998) 1–28.

[26] T. Levi-Civita, Nozione di parallelismo in una varietá qualcunque e conseguente spezificazione geometrica della curvatura Riemanniana, Rend. Circ. Mat. Palermo 42 (1917) 173–204.

[27] V.A. Mirzoyan, Classification of Ric-semiparallel hypersurfaces in Euclidean spaces, Sb. Math. 191 (2000) 1323–1338.

[28] J. Schouten, Erlangen programme und vebertragungslehre; neue gesichtspunkte zur grundlegung den geometrie, Rend. Circ. Mat. Palermo 50 (1926) 142–169.

[29] Z. Szabó, Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$. I. The local version, J. Differential Geom. 17 (1982) 531–582.

[30] Z. Szabó, Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$. II. The global version, Geom. Dedicata 19 (1985) 65–108.

[31] L. Verstraelen, Comments on the pseudo-symmetry in the sense of Deszcz, in: F. Dillen, et al. (Eds.), Geometry and Topology of Submanifolds, vol. VI, World Scientific Publ. Co., Singapore, 1994, pp. 119–209.